Title: Geometer's Sketchpad and Related Rates

Brief Overview:

Students will explore a classic related rates problem using the Geometer's Sketchpad animation. They will develop an intuitive understanding of the derivatives in the problem by examining ratios in the geometry problem. The idea of direct variation also plays an important role.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*

- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*

- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*

- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*

- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*

- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

- **Content Standards**

  Number and Operations
  Students will evaluate functions.

  Algebra
  Students will compose, manipulate, and analyze symbolic expressions, and they will examine direct relationships.

  Geometry
  Students will use properties of similar triangles and explore ratios of sides and area.

  Measurement
  Students will measure lengths of segments and use a formula for measuring areas of triangles.

- **Process Standards**

  <u>Mathematics as Problem Solving, Reasoning and Proof, Communication, Connections, and Representation</u>
  These five process standards are threads that integrate throughout the unit, although they may not be specifically addressed in the unit. They emphasize the need to help students develop the processes that are the major means for doing mathematics, thinking about mathematics, understanding mathematics, and communicating mathematics.

Links to Maryland High School Mathematics Core Learning Units:

Functions and Algebra

- **1.1.1**
  Students will recognize, describe, and extend functional relationships that are expressed algebraically and geometrically.

- **1.1.2**
  Students will represent functional relationships as a mathematical expression.

- **1.1.3**
  Students will manipulate algebraic expressions.

- **1.2.1**
  Students will determine the equation of a line, solve linear equations, and describe the solutions using numbers and symbols.

Geometry, Measurement, and Reasoning

- **2.1.1**
  Students will describe the characteristics of geometric figures and will construct and draw geometric figures using technology.

- **2.2.1**
  Students will identify similar figures and apply proportionality of their corresponding parts.

- **2.2.2**
  Students will solve problems using two-dimensional figures.

- **2.3.2**
  Students will use techniques of measurement and will calculate area of two-dimensional figures.

Grade/Level:

Calculus

Duration/Length:

Two fifty minute periods or one extended period.

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Direct variation
- Basic properties of right and similar triangles
- Implicit differentiation and related rates
- Facility with Geometer's Sketchpad

Student Outcomes:

Students will:

- gain a visual and algebraic understanding of related rates and direct variation.

Materials/Resources/Printed Materials:

- The Geometer's Sketchpad

Development/Procedures:

Students will approach a related rates problem in two ways. First, using Geometer's Sketchpad, they will create an animation to show that certain ratios of lengths and areas are constant. They will next create algebraic expressions to represent and relate the various quantities concerned. Students will then use their knowledge of implicit differentiation. Finally, they will relate the geometric and analytical approaches.

Assessment:

This is an activity-based lesson. The students, in groups, will work through two activities that require short and extended student-produced responses. A guide is attached for assessing these activities.

Extension/Follow Up:

- Students can explore other relationships in this particular problem (e.g., the rate of change of the angle of elevation from the tip of Bob's shadow to the light source).

- Students can be encouraged to visualize related rates problems, viewing them as physical/geometrical situations, not just as algebraic/analytical exercises. Students can be encouraged to use animations in Geometer's Sketchpad to present and visualize calculus problems.

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Assessment: Teacher's Guide

Introduction

Students will explore a classic related rates problem using a Geometer's Sketchpad animation. They will develop an intuitive understanding of the derivatives in the problem by examining ratios in the geometry problem. Students will approach a related rates problem in two ways. First, using Geometer's Sketchpad they will create an animation to show that certain ratios of lengths and areas are constant. They will next create algebraic expressions to represent and relate the various quantities concerned. Students will then use their knowledge of implicit differentiation and the idea of direct variation. Most importantly, the student will seek to understand and show the relationship between the geometric ratios and the derivatives.

Objectives Covered

Gain a visual and algebraic understanding of related rates.

Tools/Materials Needed for Assessment

Geometer's Sketchpad and attached activity sheets.

Administering the Assessment

Students can approach Activity 1 without being explicitly warned that they are tackling a related rates problem through geometry. On the following pages are the diagram for the animation in the first activity and its script. It is essential in their animations that Bob be constructed as a segment perpendicular to the ground and that the ray of light be constructed as a ray from the lamp and through the tip of Bob. Otherwise the light ray won't change as Bob moves. The animation requires the base point of Bob to move along the ground.

The geometric part of the Activity can be the first part of an extended period or the first of two days.

In the second Calculus part of the Activity, the students are approaching the problem as a traditional related rates exercise. The key to this activity is noting and understanding that the rate of change of the length of the shadow (and the area of the triangle) are directly related to Bob's rate. In other words, they differ by a constant (derived from Bob's height and the height of the lamppost). This comes out geometrically in the constant ratios they find in the first section.

Make Bob Walk!

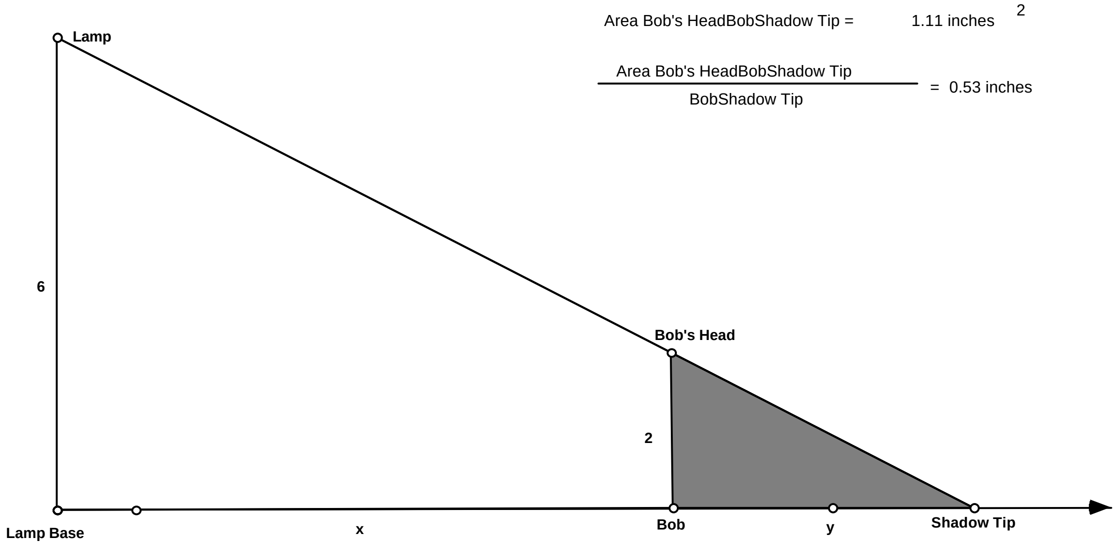
Lamp BaseBob = 4.23 inches

BobShadow Tip = 2.07 inches

$\frac{\text{Lamp BaseBob}}{\text{BobShadow Tip}} = 2.04$

Area Bob's HeadBobShadow Tip = 1.11 inches^2

$\frac{\text{Area Bob's HeadBobShadow Tip}}{\text{BobShadow Tip}} = 0.53$ inches



Script

Given:
1. Point A
2. Point B
3. Point C
4. Point D
5. Point Lamp Base
6. Point Lamp

Steps:
1. Let [j] = Ray between Point B and Point A.
2. Let [k] = Ray between Point D and Point C.
3. Let [l] = Perpendicular to Ray [k] through Point C (hidden).
4. Let Bob = Random point on Ray [k].
5. Let [m] = Perpendicular to Ray [k] through Point Bob (hidden).
6. Let [G] = Unit Point of coordinate system with Origin Lamp Base.
7. Let x = the horizontal axis (hidden).
8. Let y = the vertical axis (hidden).
9. Let Bob's Head = Random point on Line [m].
10. Let [n] = Ray between Point Bob's Head and Point Lamp (hidden).
11. Let 6 = Segment between Point Lamp Base and Point Lamp.
12. Let 2 = Segment between Point Bob's Head and Point Bob.
13. Let Shadow Tip = Intersection of Ray [k] and Ray [n].
14. Let [s] = Segment between Point Bob's Head and Point Lamp.
15. Let [t] = Segment between Point Shadow Tip and Point Bob's Head.
16. Let [u] = Segment between Point Lamp and Point Shadow Tip.
17. Let x = Segment between Point Bob and Point Lamp Base.
18. Let y = Segment between Point Shadow Tip and Point Bob.
19. Let [x] = Segment between Point Lamp Base and Point Shadow Tip.
20. Let Measurement [m1] = Length (Segment [x]).
21. Let Measurement [m2] = Distance (Lamp Base to Bob).
22. Let Measurement [m3] = Distance (Bob to Shadow Tip).
23. Let Measurement [m4] = m2/m3.
24. Let [p1] = Polygon interior with vertices Shadow Tip, Bob and Bob's Head (medium).
25. Let Measurement [m5] = Area (Polygon [p1]).
26. Let Measurement [m6] = m5/m3.

Student Activity/Assessment: Geometer's Sketchpad and Related Rates

Bob, who is 2 meters tall, walks away from a lamppost that is 6 meters tall at a constant rate of 1.5 meters per second. How fast is the length of his shadow increasing when he is 7 meters from the lamp? 11 meters? 23 meters? How fast is the ratio of the area of the triangle formed by Bob and his shadow increasing at those particular distances?

Activity 1: A Geometric Approach to the Problem

Draw the above situation using Geometer's Sketchpad. Be aware that the lamppost and Bob are perpendicular to the ground. Animate the sketch to show Bob walking away from the lamppost. Print and attach your sketch to this page.

Calculate the ratio of the distance from Bob to the lamppost, x, to the length of his shadow, y, at the distances indicated above.

Calculate the area of the triangle formed by Bob and his shadow, at the distances indicated above.

Calculate the ratio of this area to length of Bob's shadow, at the distances indicated above.

Analyze the ratios you have found. As Bob moves away from the lamppost, is the length of his shadow increasing more and more quickly, or less? Is the area increasing more and more quickly, or less?

Activity 2: A Calculus Approach to the Same Problem (Page 1)

Determine the relationship between *x* and *y*, using the triangles drawn. Express *y* in terms of *x*.

Use implicit differentiation to find a formula to show how fast the length of Bob's shadow is increasing with respect to time as Bob walks away from the lamppost. Evaluate.

How is *dy/dt* related to *dx/dt*?

Write the formula for the triangular area of Bob's shadow.

Now use the area formula and implicit differentiation to find the rate of change of the area of the triangle with respect to time, *dA/dt*. Show your derivation and evaluate.

How is *dA/dt* related to *dy/dt*, and so to *dx/dt*?

Activity 2: A Calculus Approach to the Same Problem (Page 2)

What is this kind of relationship called? What physical/geometrical situation does this relationship reflect?

Now relate these derivatives to the ratios you found in Activity 1. Explain how the ratios you found there and the derivative formulas you just found are two ways of expressing the same physical fact.

Assessment/Solutions

Scoring Guide

At the end of each section in square brackets is the point value for the question and scoring rubric as appropriate.

Student Activity/Assessment: Geometer's Sketchpad and Related Rates

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Activity 1: A Geometric Approach to the Problem

Draw the above situation using Geometer's Sketchpad. Be aware that the lamp and Bob are perpendicular to the ground. Animate the sketch to show Bob walking away from the lamppost. Print and attach your sketch to this page.

See script and sketch on previous pages.

[10: Sketch done completely; drawing to scale; animation works as directed; both sketch and script printed.
8: Sketch done, not necessarily to scale; animation partly working; both sketch and script printed.
5: Sketch done, not necessarily to scale; no animation; only one printed out.
2: Some of sketch done or incorrectly done; not to scale; no animation.
0: No response.]

Calculate the ratio of the distance from Bob to the lamppost, x, to the length of his shadow, y, at the distances indicated above.

2:1 for each of the indicated distances [1 each]

Calculate the area of the triangle formed by Bob and his shadow, at the distances indicated above.

$A=1/2*2*y=y$ [2]
$y=2x/(6-2)=x/2$, so $A=x/2$ [2]

so, $Area=7/2$ when $x=7$ [1]
$Area =11/2$ when $x=11$ [1]
$Area=23/2$ when $x=23$ [1]

Calculate the ratio of this area to length of Bob's shadow, at the distances indicated above.

Area / y = y / y = 1/1 *[2]*
So, the ratio is 1:1 at each of the distances. *[1 each]*

Analyze the ratios you have found. As Bob moves away from the lamppost, is the length of his shadow increasing more and more quickly, or less? Is the area increasing more and more quickly, or less?

Since both ratios are constant at all values, the length of Bob's shadow is increasing at a constant rate, as is the area.

[5: Correct answers with clear explanation in complete sentences relating constant ratios to constant rates of increase.
3: Correct answers but partial or incorrect explanation.
1: Correct answers but no explanation.
0: Incorrect or no answer.]

Activity 2: A Calculus Approach to the Same Problem (Page 1)

Determine the relationship between *x* and *y*, using the triangles drawn. Express *y* in terms of *x*.

$$\frac{y}{2} = \frac{x+y}{6}$$

$$y = \frac{x}{2}$$

[3: Writing the relating equation and simplifying correctly.
2: Writing the relating equation and not simplifying correctly.
1: Incorrect equation.
0: No response.]

Use implicit differentiation to find a formula to show how fast the length of Bob's shadow is increasing with respect to time as Bob walks away from the lamppost. Evaluate.

$$\frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{1}{2}(1.5m/s) = .75m/s$$

[1 for correct differentiation and 1 for correct evaluation.]

How is *dy/dt* related to *dx/dt*?

They are directly related to each other. [2]

Write the formula for the triangular area of Bob's shadow.

A = ½ (2)y = y [1 for formula and 1 for simplification.]

Now use the area formula and implicit differentiation to find the rate of change of the area of the triangle with respect to time, *dA/dt*. Show your derivation and evaluate.

$$\frac{dA}{dt} = \frac{dy}{dt} \qquad \frac{dA}{dt} = .75m^2/s$$

[1 for correct differentiation and 1 for correct evaluation.]

How *is dA/dt* related to *dy/dt*, and so to *dx/dt*?

dA/dt is directly related to dy/dt, and therefore by transitivity is directly related to dx/dt. [2]

Activity 2: A Calculus Approach to the Same Problem (Page 2)

What is this kind of relationship called? What physical/geometrical situation does this relationship reflect?

This is a ***direct*** *relationship. This relationship is reflected by the fact the triangles are similar triangles.*

[1 point for stating that it is a direct relationship and 1 point for explaining why.]

Now relate these derivatives to the ratios you found in Activity 1. Explain how the ratios you found there and the derivative formulas you just found are two ways of expressing the same physical fact.

The physical fact is that as Bob moves from the lamppost at a constant rate, the length of his shadow increases at a constant rate. The geometric manifestation of this is that the two triangles formed will always be similar, and therefore the stated ratios are constant. The Calculus manifestation of this is that the derivatives are directly related to each other, i.e., differ by a constant.

[6: Full answer relating physical situation, geometric ratios, and derivatives correctly. Students use complete sentences.
4: Students make a connection between geometry and Calculus but explanation is incomplete or not fully correct.
2: Only refers to geometry or Calculus. Incomplete sentences.
0: No response.]